

Auctions with Financial Externalities*

Emiel Maasland[†] and Sander Onderstal[‡]

June 29, 2005

Abstract

We study auctions with financial externalities, i.e., auctions in which losers care about how much the winner pays. In the first-price auction, larger financial externalities result in a lower expected price; in the second-price auction, the effect is ambiguous. Although the expected price in the second-price auction may increase if financial externalities increase, the seller is not able to gain more revenue by guaranteeing the losers a fraction of the auction revenue. With a reserve price, we find that both auctions may have pooling at the reserve price. This finding suggests that identical bids need not be a signal of collusion, in contrast to what is sometimes argued in anti-trust cases.

Keywords and Phrases: Auctions, financial externalities, reserve price.

JEL Classification Numbers: D44, C72

*For valuable discussions and comments, we would like to thank Eric van Damme, Jacob Goeree, Thomas Kittsteiner, Marta Kolodziejczyk, seminar participants at Tilburg University, Humboldt University Berlin, and National University of Singapore, and audiences at ESEM 2001 in Lausanne, and the FEEM 2002 conference in Milan on auctions and market design. The suggestions of an anonymous referee of this Journal greatly improved the article. The usual disclaimer applies.

[†]SEOR, Erasmus University, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands, +31-10-408-1513, fax +31-10-408-9650, emaasland@few.eur.nl, www.seor.nl/ecri/staff/emiel.html.

[‡]University of Amsterdam, Economics Department, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands, +31-20-525-7161, fax +31-20-525-5591, onderstal@uva.nl, www.fee.uva.nl/pp/onderstal. Onderstal gratefully acknowledges financial support from the Dutch National Science Foundation (NWO 510.010.501 and NWO-VICI 453.03.606).

1 Introduction

In this paper, we study sealed-bid auctions with financial externalities. Financial externalities arise when losers benefit directly or indirectly from a high price paid by the winner(s). In auction theory, it is generally assumed that losers are indifferent about how much the winner(s) pay(s). However, in real life auctions, this assumption might be false. In reality, an auction is not an isolated game, as winners and losers also interact after the auction. Paying a high price in the auction could make a winner a weaker competitor later.

The series of UMTS auctions¹ that took place in Europe offers a concrete example of auctions in which losers benefit *indirectly* from a high price paid by the winners. In this context, there are at least three ways that firms that do not acquire a license may benefit from a winning firm that pays a high price. First, the share values of winning firms may drop, which makes the winner vulnerable to a hostile take-over by competing firms. For instance, the drop in share value of the Dutch telecom company KPN of about 95% is partly explained by the huge sum of money the company spent to acquire British, Dutch and German UMTS licences.^{2,3} Second, if firms are budget constrained, a high payment in the first auction may give competing firms an advantage in later auctions. Third, high payments may force the winning firms to cut their budget for investment, which may be favorable for the losers' position in the telecommunications market, as the losing firms are not only competitors of the winning firms in the auction, but in the telecommunications market as well. Indeed, Börgers and Dustmann (forthcoming) argue that financial externalities may partly explain seemingly irrational bidding in the British UMTS auction.

Financial externalities occur *directly* when losing bidders receive money from the winner(s). For instance, this may happen in the case of bidding rings, in which a member of the ring receives money when she does not win the object. Also, partnerships are dissolved

¹These are spectrum auctions. UMTS stands for Universal Mobile Telecommunication Services: a third generation mobile telecommunication standard.

²In the UK, KPN bought part of the TIW license after the auction. In Germany, KPN has a majority share in E-Plus.

³The other part of the drop is probably explained by changed sentiment in the market.

using an auction in which losing partners obtain part of the winner's bid. In takeover contests, losing bidders who have shares in the target firm receive payoffs proportional to the sales price. Furthermore, the owner of a large estate may specify in his last will that after his death, the estate should be sold to one of the heirs in an auction, where the auction revenue is divided among the losers. Finally, in the 'Amsterdam auction', the loser receives a premium proportional to the difference between her bid and the minimum price (Goeree and Offerman, 2004).

In Section 2, we present a model of bidding in sealed-bid auctions with financial externalities. Either the first-price sealed-bid auction (FPSB) or the second-price sealed-bid auction (SPSB) is used to sell an indivisible object. We assume an independent private signals model, with private values and common value models as special cases. Financial externalities are exogenously given and modelled by a parameter φ that is inserted in the bidders' utility functions. This is the simplest extension of the independent private signals model which incorporates financial externalities. Despite its admitted simplicity, this model appears to be sufficiently rich to generate interesting insights.

In Section 3, we derive results for FPSB and SPSB without reserve price. We find a unique symmetric and efficient bid equilibrium for each of the two auction types. Equilibrium bids in FPSB decrease when φ increases. An intuition for this result is that larger financial externalities make losing more attractive for the bidders so that they submit lower bids. The effect of financial externalities on the equilibrium bids in SPSB is ambiguous. A possible explanation is that in SPSB, a bidder is not only inclined to bid less the higher φ is (as she gets positive utility from losing), she also has an incentive to bid higher, because, given that she loses, she is able to influence directly the level of payments made by the winner. Moreover, we construct an example in which the seller's revenue increases when φ increases. This finding suggests that the seller may gain more revenue by guaranteeing the losers a fraction of the auction revenue. This, however, turns out not to be the case: under no circumstances does revenue sharing increase revenue. Finally, we give a revenue comparison between FPSB and SPSB. We find that SPSB results in a higher expected revenue than FPSB unless a bidder's interest in her own payments is equal to the sum of the other bidders' interest in what she pays. In that case, FPSB and

SPSB are revenue equivalent.

In Section 4, we characterize equilibrium bid strategies for the case that a reserve price is imposed in FPSB and SPSB. For simplicity, we assume a model with independent private values. In this section, we introduce the concept of a weakly separating Bayesian Nash equilibrium, which is an equilibrium in which all types below a threshold type abstain from bidding, and all types above this type submit a bid according to a strictly increasing bid function. We find that FPSB has no weakly separating Bayesian Nash equilibrium. However, we derive an equilibrium in which bidders with low signals abstain from bidding, bidders with intermediate signals pool at the reserve price, and bidders with high signals submit a bid according to a strictly increasing bid function. SPSB has a weakly separating Bayesian Nash equilibrium if, and only if, the reserve price is sufficiently low (or so high that no bidder submits a bid). Otherwise, the equilibrium involves pooling at the reserve price.

These findings may shed new light on observations of identical bids in auctions. Theoretical work suggests that several bidders submitting a bid equal to the reserve price is a signal of collusive behavior (McAfee and McMillan, 1992). In public policy, there is also general acceptance of the view that identical bids suggest the presence of an agreement. Comanor and Schankerman (1976) report that competition authorities in the US required auctioneers to report instances of identical bids. In judicial judgement, this view is generally accepted too. A good example can be found in *U.S. v. Pfizer*, where the court stated: “Certainly, uniformity of price may be and has been considered some evidence tending to establish an illegal agreement”.⁴ Our finding indicates that several bidders submitting a bid equal to the reserve price may not be explained by collusion but by the existence of financial externalities.

In the next subsection, we discuss the related literature.

1.1 Related literature

Our paper is related to a large range of papers in finance, industrial organization, and micro-economics that study similar models. In order to better link our paper to this

⁴*United States v. Chas. Pfizer Co., Inc.*, 217 F. Supp. 199 (1963) at 201.

literature, we anticipate some of the specifics of our model.⁵ We define the utility function of each bidder i as follows:

$$u_i(j, b) = \begin{cases} v_i - b & \text{if } j = i \\ \varphi b & \text{if } j \neq i, \end{cases} \quad (1)$$

where v_i is the value that i attaches to the auctioned object, j is the winner of the auction, b is the payment by the winner, and $\varphi > 0$ is the parameter indicating financial externalities.

Engers and McManus (2004) investigate a related model in the context of charity auctions, employing the following payoff structure:

$$u_i(j, b) = \begin{cases} v_i - b + \theta b & \text{if } j = i \\ \lambda b & \text{if } j \neq i, \end{cases} \quad (2)$$

The parameter $\lambda > 0$ can be interpreted as an altruistic feeling a person obtains if another bidder wins the auction (as the auction revenue is transferred to a charitable organization). $\theta \geq \lambda$ represents the utility the winner gets for her own contribution, where $\theta - \lambda$ is interpreted as the winner's 'warm glow' for giving to charity. Engers and McManus' model is a special case of ours in the sense that in the absence of a reserve price, the equilibrium bids in Engers and McManus can be constructed from the equilibrium bids in our framework. To see this, rewrite (2) as

$$u_i(j, b) = \tilde{u}_i(j, \tilde{b}) = \begin{cases} v_i - \tilde{b} & \text{if } j = i \\ \varphi \tilde{b} & \text{if } j \neq i, \end{cases}$$

where

$$\begin{aligned} \tilde{b} &= (1 - \theta)b, \text{ and} \\ \varphi &= \frac{\lambda}{1 - \theta} \end{aligned}$$

so that our model applies with the utility function \tilde{u} . Observe that a strictly positive reserve price may imply that the candidate for the equilibrium bid in our model is below the reserve price, as this bid is constructed by multiplying the equilibrium bid in Engers and McManus by $1 - \theta < 1$. Engers and McManus study their model in an independent

⁵We are grateful to an anonymous referee for suggesting this set-up.

private value setting, while we focus on a more general setting with independent private signals.

Quite some papers in the literature study special cases of Engers and McManus' model, and hence special cases of ours.⁶ For instance, Engelbrecht-Wiggans (1994), Bulow et al. (1999), Ettinger (2003c), and Goeree et al. (forthcoming) assume $\lambda = \theta$, and apply this model to a large range of economic settings, including charity auctions and take-over battles in which the bidding firms own a toehold in the target. Engelbrecht-Wiggans (1994) analyzes a model with affiliated signals, while Bulow et al. (1999), Ettinger (2003c) and Goeree et al. (forthcoming) study special cases of this model: Bulow et al. (1999) focus on common values, Ettinger (2003c) and Goeree et al. (forthcoming) on independent private values. Engelbrecht-Wiggans (1994) and Engers and McManus (2004) show that SPSB generates more revenue than FPSB, a result we show to hold true in our context as well. In contrast, Bulow et al. (1999) prove that small asymmetries in the toehold may imply that FPSB is superior to SPSB in terms of revenues raised. Engers and McManus (2004) and Goeree et al. (forthcoming) show that all-pay auctions dominate winner-pay auctions (such as FPSB and SPSB) as in the latter auctions, bidders forgo a positive externality if they top another's high bid.

A specific interpretation of our model is a situation in which all losing bidders equally share the payment by the winner, i.e., $\varphi = 1/(n - 1)$, where n is the total number of bidders. This may be the case in situations of 'knock-out' auctions organized by bidding rings (Graham and Marshall, 1987, McAfee and McMillan, 1992, and Deltas, 2002), dissolving partnerships (Cramton et al., 1987, Angeles de Frutos, 2000, Kittsteiner, 2003, and Morgan, 2004), and heirs bidding for a family estate (Engelbrecht-Wiggans, 1994). If $n = 2$ and $\varphi = 1$, then FPSB and SPSB are special cases of the k -double auction with $k = 0$ and $k = 1$ respectively (Van Damme, 1992 and Kittsteiner, 2003).⁷

We add the following to the above mentioned papers. First, we derive comparative statics that are different from the ones in the other models. For instance, Engelbrecht-

⁶Ettinger (2003a) studies a more general model allowing for non-linear financial externalities, and compares FPSB and SPSB in a model with complete information.

⁷The k -double auction has the following rules. Both bidders submit a bid. The highest bidder wins the object, and pays the loser an amount equal to $kb_L + (1 - k)b_W$, where b_L is the loser's bid, b_W the winner's bid, and $k \in [0, 1]$.

Wiggans (1994) shows that the equilibrium bid functions of FPSB and SPSB are increasing in θ if $\lambda = \theta$. In our model, the effect of φ on the equilibrium bids can be both increasing and decreasing.⁸ Second, we examine the question whether the seller has an incentive to share a fraction of the auction revenue with the losing bidders. Bulow et al. (1999) show in their setting that the seller may increase the revenue of SPSB by giving a weak bidder a toehold, in contrast to what we find in our model. Third, we analyze the effect of the reserve price on the equilibrium bids, which, as mentioned, may be different than in the above related settings since the one-to-one relationship between the equilibrium bid functions ceases to hold. Ettinger (2003c) is one of the few papers that studies auctions with a positive reserve price. In his framework, he finds a weakly separating equilibrium for the ascending auction, while we observe pooling at the reserve price in our setting for FPSB and SPSB.

There are several other papers related to ours. Jehiel and Moldovanu (1996, 2000), Jehiel et al. (1996, 1999), and Das Varma (2002) study auctions in which losing bidders receive positive or negative allocative externalities from the winner. More specifically, they study the following model:

$$u_i(j, b) = \begin{cases} v_i - b & \text{if } j = i \\ \alpha_{ji} & \text{if } j \neq i, \end{cases}$$

where α_{ji} is the externality bidder j imposes on bidder i ($\alpha_{ji} > 0$ [$\alpha_{ji} < 0$] if the externality is positive [negative]). Since the utility of the bidders in these articles is affected by the identity of the winner and not by how much the winner pays, these externalities are clearly different from financial externalities. Jehiel and Moldovanu (2000) derive equilibrium bid strategies that involve some pooling at the reserve price for SPSB with positive externalities. This equilibrium structure is similar to the one we find in the case of financial externalities.

Another set of papers focuses on situations in which bidding firms own shares in each other (Ettinger, 2003b, and Dasgupta and Tsui, 2004). Firm i is controlled by shareholder i , who also has a minority share in some of the other firms. The utility structure of these

⁸Intuitively, if both the winner and the losers profit from the payment by the winner, bidders do have more incentive to bid higher than if only losers profit. This explains why the effect of θ and φ can be opposite.

models is

$$u_i(j, b) = (v_j - b) \sigma_{ij}$$

where σ_{ij} denotes the share of shareholder i in firm j . According to Dasgupta and Tsui (2004), the seller expects more revenue in FPSB than in SPSB for the case of two symmetric players, which is contrary to what we find.

Finally, Benoît and Krishna (2001) study a two-bidder model with complete information in which two objects are sold sequentially. As bidders are budget constrained, a particular bidder's payoff is affected by the price paid by a rival bidder, so that their model can be interpreted as a model with endogenously determined financial externalities.

2 The model

We consider a situation with $n \geq 2$ risk neutral bidders, numbered $1, 2, \dots, n$, who bid for one indivisible object. The auction being used is either FPSB or SPSB. Each of these two auction types may or may not have a reserve price.

Essentially, we use Milgrom and Weber's (1982) model with independent signals instead of affiliated signals as a starting point. We assume that each bidder i receives a one-dimensional private signal t_i (we also say that bidder i is of type t_i). We will let $v_i(\mathbf{t})$ denote the value of the object for bidder i given the vector $\mathbf{t} \equiv (t_1, \dots, t_n)$ of all signals. Special cases are independent private value models ($v_i(\mathbf{t})$ only depends on t_i), and common value models ($v_i(\mathbf{t}) = v_j(\mathbf{t})$ for all i, j, \mathbf{t}). Without loss of generality, we assume that the signals t_i are independently drawn from the uniform distribution on the interval $[0, 1]$.⁹

We make the following assumptions on the functions v_i .

Value Differentiability: v_i is differentiable in all its arguments, for all i, \mathbf{t} .

Value Monotonicity: $v_i(\mathbf{t}) \geq 0$, $\frac{\partial v_i(\mathbf{t})}{\partial t_i} > 0$, $\frac{\partial v_i(\mathbf{t})}{\partial t_j} \geq 0$, and $\frac{\partial v_i(\mathbf{t})}{\partial t_i} > \frac{\partial v_i(\mathbf{t})}{\partial t_j}$ for all i, j, \mathbf{t} .

Symmetry: $v_i(\dots, t_i, \dots, t_j, \dots) = v_j(\dots, t_j, \dots, t_i, \dots)$ for all t_i, t_j, i, j .

⁹Suppose the signals t_i are drawn from strictly increasing distribution functions F_i . Such a model is isomorphic to a model with uniformly distributed signals \tilde{t}_i and value functions \tilde{v}_i , where $\tilde{t}_i \equiv F_i(t_i)$ and $\tilde{v}_i(\tilde{t}_1, \dots, \tilde{t}_n) \equiv v_i(F_1^{-1}(\tilde{t}_1), \dots, F_n^{-1}(\tilde{t}_n))$.

Value Differentiability is imposed to make the calculations on the equilibria tractable. *Value Monotonicity* indicates that all bidders are serious and that bidders' values are strictly increasing in their own signal and weakly in the signals of the others. Moreover, it includes a single crossing property. *Symmetry* may be crucial for the existence of efficient equilibria in standard auctions. *Value Differentiability*, *Value Monotonicity*, and *Symmetry* together ensure that the bidder with the highest signal is also the bidder with the highest value. These assumptions therefore imply that the seller assigns the object efficiently if and only if the bidder with the highest signal gets it.

Also, let us define $v(x, y)$ as the expected value that bidder i assigns to the object, given that her signal is x , and that the highest signal of all the other bidders is equal to y :

$$v(x, y) \equiv E\{v_i(\mathbf{t}) | t_i = x, \max_{j \neq i} t_j = y\}.$$

By *Symmetry*, v does not depend on i .

The bidders are expected utility maximizers. Each bidder is risk neutral and cares about what other bidders pay in the auction. The utility of the bidders is specified in (1). It is a natural assumption to let a bidder's interest in her own payments be larger than or equal to sum of the other bidders' interest in her payments, so that we assume $\varphi \leq 1/(n - 1)$.

3 Zero reserve price

Consider FPSB and SPSB with a zero reserve price. In the case of direct financial externalities, the seller does not receive the entire bid of the winner. Therefore, we distinguish between the seller's expected revenues and the expected price, i.e. the total payment by the winner. Of course, in the case of indirect financial externalities, the seller's expected revenues and the expected price are the same.

3.1 First-price sealed-bid auction

The following proposition characterizes the equilibrium bid function for FPSB.¹⁰ To derive equilibrium bidding, we suppose that, in equilibrium, all bidders use the same bid function. By a standard argument, this bid function must be strictly increasing and continuous. Let $U(t, s)$ be the utility for a bidder with signal t who behaves as if having signal s , whereas the other bidders play according to the equilibrium bid function. A necessary equilibrium condition is that

$$\frac{\partial U(t, s)}{\partial s} = 0$$

at $s = t$. From this condition, a differential equation can be derived from which the equilibrium bid function is uniquely determined. The auction outcome is efficient. Observe that in the case of independent private values ($v(x, y)$ only depends on x), the bid function is strictly increasing in n .

Proposition 1 *The unique Bayesian Nash equilibrium of FPSB is characterized by*

$$B_1(\varphi, t) = v(t, t) - \frac{\varphi}{1 + \varphi} v(t, t) - \frac{1}{1 + \varphi} \int_0^t \frac{dv(y, y)}{dy} \left(\frac{y}{t}\right)^{(n-1)(1+\varphi)} dy, \quad (3)$$

where $B_1(\varphi, t)$ is the bid of a bidder with signal t . The outcome of this auction is efficient.

Each of the terms of the right-hand side (RHS) of (3) has an attractive interpretation. The first term is the equilibrium bid for a bidder with type t in SPSB without financial externalities, as in the absence of financial externalities, in SPSB, a bidder will submit a bid equal to her maximal willingness to pay given that her strongest opponent has the same signal as she (Milgrom and Weber, 1982). The second term is the bid shading that would occur if all bidders attached the same value $v(t, t)$ to the object: in such a situation, if a bidder wins at a bid of b , her utility is $v(t, t) - b$, while if an opponent wins at the same bid, her utility is φb . Equating these utilities results in a bid of $\frac{1}{1+\varphi} v(t, t)$. The third

¹⁰The proof of this and all other propositions are relegated to the Appendix.

term can be interpreted as the strategic bid shading because of private values. Note that if $\varphi = 0$, this term is equal to the standard strategic bid shading in FPSB.

This interpretation of the equilibrium bid function suggests that this function is decreasing in φ , which in fact follows directly from (3):

Corollary 1 *Increasing φ decreases $B_1(\varphi, t)$.*

From Corollary 1, it immediately follows that the expected price is decreasing in φ .

Corollary 2 *Increasing φ decreases the expected price the winner pays in FPSB.*

Observe that Corollary 2 implies that the seller's revenue is decreasing in φ , both for direct and indirect financial externalities. The seller's revenue in the case of direct financial externalities is a fraction $(1 - (n - 1)\varphi)$ of the revenues under indirect financial externalities. Corollary 2 indicates that revenue under indirect financial externalities is a decreasing function of φ , which also holds true for direct financial externalities as $(1 - (n - 1)\varphi)$ is decreasing in φ as well. Therefore, it is not attractive for the seller to have the losers share a fraction of the auction revenue.

Corollary 3 *The seller cannot increase his revenue from FPSB by guaranteeing the losing bidders a share of the auction revenue.*

3.2 Second-price sealed-bid auction

Equilibrium bids for SPSB are obtained using the same logic as for FPSB. The analysis reveals uniqueness and efficiency of the equilibrium bid function. Observe that in the case of private values, the bid function does not depend on n .¹¹

¹¹This is actually a quite subtle observation, as n does not appear in the expression for the equilibrium bid. However, in general, $v(t, t)$ depends on n .

Proposition 2 *The unique Bayesian Nash equilibrium of SPSB is characterized by*

$$B_2(\varphi, t) = v(t, t) - \frac{\varphi}{1 + \varphi}v(t, t) + \frac{\varphi}{(1 + \varphi)(1 + 2\varphi)} \int_t^1 \frac{dv(y, y)}{dy} \left(\frac{1 - y}{1 - t} \right)^{\frac{1 + \varphi}{\varphi}} dy \quad (4)$$

where $B_2(\varphi, t)$ is the bid of a bidder with signal t . The outcome of this auction is efficient.

Each term of the RHS of (4) has its attractive interpretation. From the discussion of FPSB, it follows that the first term is the bid in SPSB in the absence of financial externalities, and the second term is the bid shading in the hypothetical situation that all bidders attach the same value $v(t, t)$ to the object. The third term increases the bid as a bidder of type t has an incentive to drive up the payment by types above t .

In contrast to FPSB, the effect of an increase in φ on the equilibrium bids in SPSB is dependent on a bidder's type. From (4), it is clear that the equilibrium bid of the highest type is decreasing in φ . The reason is that as this bidder does not have a type above her, she does not have an incentive to drive up the price. However, the effect of φ on the equilibrium bids of the other types is not clear. The effect of the second term of the RHS of (4) may be larger as well as smaller than the third term. The following example illustrates how equilibrium bidding is affected when φ is varied.

Example 1 (Effect of φ on equilibrium bidding) *Let $v(t, t) = t$ for all $t \in [0, 1]$. The equilibrium bid function is given by*

$$B_2(\varphi, t) = \frac{\varphi}{(1 + \varphi)(1 + 2\varphi)} + \frac{1}{1 + 2\varphi}t, \quad t \in [0, 1].$$

As B_2 is a continuous function in both φ and t , the following can be derived. First, there is a strictly positive mass of types close to zero for which the effect of φ is ambiguous in

the sense that for φ close to 0, an increase in φ leads to higher bids and for φ close to 1, an increase in φ leads to lower bids. This follows from the following observations.

$$\frac{\partial B_2(0, 0)}{\partial \varphi} = 1 > 0,$$

and

$$\frac{\partial B_2(1, 0)}{\partial \varphi} = -\frac{1}{36} < 0.$$

Intuitively, if φ is large enough, $B_2(\varphi, t)$ decreases as for each bidder, losing becomes more interesting due to higher financial externalities. Second, the equilibrium bids of types close to 1 are decreasing in φ . This follows from the fact that $B_2(\varphi, 1) = \frac{1}{1+\varphi}$.

Also, the effect of φ on the expected price paid by the winner is ambiguous. This follows from Example 2, in which the expected price is increasing if φ is small, and decreasing if φ is large.

Example 2 (Effect of φ on the expected price) Let $v(t, t) = t$ and $n = 2$. The expected price is equal to the expectation of $B_2(\varphi, t^{(2)})$ with respect to the second highest signal $t^{(2)}$, which is given by

$$E_{t^{(2)}}\{B_2(\varphi, t^{(2)})\} = \frac{1 + 4\varphi}{3(1 + \varphi)(1 + 2\varphi)}.$$

This continuous function is increasing for φ close to 0 and decreasing for φ close to 1, as

$$\frac{\partial E_{t^{(2)}}\{B_2(0, t^{(2)})\}}{\partial \varphi} = \frac{1}{3} > 0,$$

and

$$\frac{\partial E_{t^{(2)}}\{B_2(1, t^{(2)})\}}{\partial \varphi} = -\frac{11}{108} < 0.$$

Example 2 shows that for small φ , the expected price paid by the winner may increase if φ increases, which suggests that the seller might gain more revenue by guaranteeing the losers a fraction of the payment he receives from the winner. This, however, turns out not

to be the case. We prove this proposition by using the famous revenue equivalence theorem (Myerson, 1981), which states that the expected utility of the lowest type is a sufficient statistic for the ranking of efficient auctions: the higher the utility of the lowest type, the lower the expected revenue.¹² First, consider the standard case in which the bidders obtain no indirect financial externalities. If the seller pays the losers a fraction of the auction revenue, the lowest type's expected utility goes up from zero to a strictly positive number. The reason is that the lowest type gets a fraction of the second highest bid, which is strictly positive (see Proposition 2). As SPSB is efficient, the seller's expected revenue decreases. This result turns out to remain valid if the bidders do experience financial externalities, indirect or direct (e.g., the bidders own a share in the seller), as the lowest type's utility is strictly increasing in φ .

Proposition 3 *The seller cannot increase his revenue from SPSB by guaranteeing the losing bidders a share of the auction revenue.*

3.3 Revenue comparison

Let us compare the expected revenue from FPSB and SPSB. Observe that the revenue ranking of the two auctions is the same for direct and indirect financial externalities, since the revenue in the case of direct externalities is a fraction $(1 - (n - 1)\varphi)$ of the revenue under indirect externalities. As said, the auction for which the lowest type obtains the highest expected utility generates the lowest expected revenue. It turns out that if $0 < \varphi < \frac{1}{n-1}$, SPSB generates a strictly higher expected revenue than FPSB.¹³ For $\varphi = \frac{1}{n-1}$, both auctions are revenue equivalent, which follows as the utility of the lowest type is the same for both auctions.

¹²Maasland and Onderstal (2002) show that the revenue equivalence theorem remains valid in the case that financial externalities are introduced.

¹³Engelbrecht-Wiggans (1994) claims the same result, but his proof is not correct. Engers and McManus (2004) derive this result in an independent private values model that is a special case of ours.

Proposition 4 For $\varphi < \frac{1}{n-1}$, SPSB generates a strictly higher expected revenue than FPSB. For $\varphi = \frac{1}{n-1}$, FPSB and SPSB are revenue equivalent.

Intuitively, in SPSB, a bidder can increase the payment by the winner by submitting a higher bid, which is not the case for FPSB. However, this argument does not hold true if $\varphi = \frac{1}{n-1}$. Of course, when the financial externalities are direct, the winner's payment is entirely distributed among the losers, so that the seller's revenue in both auctions is zero. Note that this implies that the expected utility of the bidders is the same for both auctions because both auctions have an efficient allocation of the object. As bidding is not affected by whether the financial externalities are direct or indirect, FPSB and SPSB yield equal expected utility to the bidders in the case of indirect financial externalities. Then it immediately follows that also for indirect financial externalities, FPSB and SPSB are revenue equivalent.

4 Positive reserve price

Consider FPSB and SPSB with a reserve price $R > 0$. In order to keep the model tractable, we assume that the independent private values model holds. With some abuse of notation, we write $v_i(\mathbf{t}) = v(t_i)$ for all i, \mathbf{t} , where v is a strictly increasing function.

This section focuses, among other things, on the existence of *weakly separating Bayesian Nash equilibria*, for which the following definition applies.

Definition 1 A *weakly separating Bayesian Nash equilibrium* is a Bayesian Nash equilibrium in which all types below a threshold type abstain from bidding, and all types above this type submit a bid according to a strictly increasing bid function.

4.1 First-price sealed-bid auction

FPSB has a symmetric equilibrium that involves pooling at the reserve price. We assume that $R < v(1)$, as otherwise, none of the bidders has an incentive to submit a bid as their

value would not be higher than the reserve price. Proposition 5 describes a Bayesian Nash equilibrium in which bidders with a type below a threshold type L do not bid, types in an interval $[L, H]$ submit a bid equal to R , and bidders with a type t above the threshold type H bid $g^R(t)$, where g^R is a strictly increasing function, defined as

$$g^R(t) \equiv (n-1)t^{-(n-1)(1+\varphi)} \left[\int_H^t y^{(n-1)\varphi+n-2} v(y) dy + v(H)H^{(n-1)(1+\varphi)} \right].$$

More specifically, let

$$H = \min\{1, v^{-1}((1+\varphi)R)\} \quad (5)$$

and L be the unique solution of

$$\varphi R (p(H, L) - L^{n-1}) = p(H, L) [v(L) - R], \quad (6)$$

where

$$p(L, H) = \frac{1}{n(H-L)} (H^n - L^n) \quad (7)$$

is the probability that a bidder wins given that she bids the reserve price.

For type H , the indifference relation (5) follows as she is indifferent between bidding R and submitting a bid just above R . These two bids result in a different outcome if H , when bidding R , loses against another bidder who also bids R . A bid just above R gives her utility $v(H) - R$, while bidding R results in φR .

The indifference relation (6) for type L is constructed as follows. L is indifferent between bidding R and abstaining from bidding. The outcome is different in two situations: first, if the bidder wins against another bidder who bids R (which occurs with probability $p(L, H) - L^{n-1}$), and if no other bidder bids (which happens with probability L^{n-1}). Note that $v(L) \geq R$: a bidder with a value below R will never enter the auction, as by entering, she increases her probability to win the object, which gives a negative utility, while potentially forgoing a positive pay-off when the highest of the bids of the other bidders is R .

Proposition 5 describes an equilibrium for the following reasons. First, L is indifferent between abstaining from bidding, and submitting a bid equal to the reserve price. Second, H is indifferent between bidding R (and therefore pool with all types in the interval

$[L, H]$), and bidding marginally higher than R . Third, the incentive compatibility constraint for types above H results in the same differential equation as the bid function for FPSB without reserve price, of which g^R is the solution satisfying the boundary condition $g^R(H) = R$.

Proposition 5 *Assume independent private values and $R < v(1)$. Let $B_1^R(\varphi, t)$, the bid of a bidder with value t , be given by*

$$B_1^R(\varphi, t) = \begin{cases} g^R(t) & \text{if } t > H \\ R & \text{if } L \leq t \leq H \\ \text{"no bid"} & \text{if } t < L \end{cases}$$

where H and L follow from (5) and (6) respectively. Then $B_1^R(\varphi, t)$ constitutes a symmetric Bayesian Nash equilibrium of FPSB if $R > 0$.¹⁴

To get an intuition why pooling at R occurs in equilibrium, consider a situation in which $n = 2$, $v(t) = t$, and $1 > R \geq \frac{1}{1+\varphi}$. In that case, $H = 1$, so that bidders either abstain from bidding, or bid R . To see why this happens in equilibrium, suppose that one of the two bidders submits a bid $b = R$ if her type exceeds L , and abstains from bidding otherwise. It could be a best response for the other bidder to bid R as well if her value is high enough. To see this, note that if she loses, then her utility is

$$\varphi R \geq \frac{\varphi}{1+\varphi} \geq \frac{\varphi t}{1+\varphi},$$

whereas winning gives her a utility of at most

$$t - R \leq t - \frac{1}{1+\varphi} \leq t - \frac{t}{1+\varphi} = \frac{\varphi t}{1+\varphi}.$$

Low types are then willing to lose the opportunity of getting the object by abstaining from bidding. High types bid R , assuring themselves the object if the other bidder does not bid, but also making sure that if the other bids, to lose as often as possible.

¹⁴Note that $B_1^R(\varphi, t)$ is continuous at H . This must be the case in equilibrium. Suppose, on the contrary, that the bid function has a jump at H . Then a bidder with a type slightly higher than H has an incentive to deviate from the bid strategy to a bid just above R .

In contrast to a situation without financial externalities, there exists no weakly separating Bayesian Nash equilibrium for FPSB:

Proposition 6 *Assume independent private values. FPSB has no weakly separating Bayesian Nash equilibrium if $R > 0$ and $\varphi > 0$.*

The proof of Proposition 6 is by contradiction. If a weakly separating Bayesian Nash equilibrium existed, all types exceeding $v^{-1}(R)$ (the type for whom the object is worth R) would submit a bid according to a strictly increasing equilibrium bid function. This function can be constructed in a similar way as the equilibrium bid function for FPSB without reserve price. A contradiction is established, as any strictly increasing equilibrium bid function requires a bidder with type $v^{-1}(R)$ to submit a bid below the reserve price.

4.2 Second-price sealed-bid auction

The shape of the equilibrium of SPSB when the seller imposes a reserve price $R > 0$ depends on the level of R . Let us start by observing that, regardless of R , all bidders who submit a bid above the reserve price do so according to the same bid function as in the absence of a reserve price. This implies that, in contrast to FPSB, SPSB has a weakly separating Bayesian Nash equilibrium given that the reserve price is not too high. This observation follows trivially when the reserve price is smaller than the lowest submitted equilibrium bid, which is strictly positive according to Proposition 2. However, in nontrivial cases weakly separating Bayesian Nash equilibria also exist. According to Proposition 7, for low R , types up to a threshold type \hat{t} abstain from bidding, and types above \hat{t} submit the same bid as in the case of no reserve price.

Proposition 7 *Assume independent private values. SPSB with reserve price R has a weakly separating Bayesian Nash equilibrium if and only if $R \leq B_2(\varphi, v^{-1}(R))$. If such*

an equilibrium exists, then it is given by:

$$B_2^R(\varphi, t) = \begin{cases} B_2(\varphi, t) & \text{if } t \geq \hat{t} \\ \text{"no bid"} & \text{if } t < \hat{t} \end{cases}$$

where $B_2^R(\varphi, t)$ is the bid of a bidder with value t . If $R < B_2(\varphi, 0)$, then $\hat{t} = 0$, otherwise \hat{t} is the unique solution of

$$\varphi B_2(\varphi, \hat{t})(1 - \hat{t}^n) + \hat{t}^n [v(\hat{t}) - R] = \varphi(1 - \hat{t})R. \quad (8)$$

The threshold type \hat{t} is indifferent between bidding $B_2(\varphi, \hat{t})$ and abstaining from bidding, and hence follows from equation (8).

An intuition for the condition $R \leq B_2(\varphi, v^{-1}(R))$ being necessary is the following. In a weakly separating Bayesian Nash equilibrium, a bidder with type R is always prepared to submit a bid of at least R . To see this, observe that for this bidder, in a weakly separating Bayesian Nash equilibrium, a bid equal to R yields the same revenue as abstaining from bidding. However, in equilibrium, each type that submits a bid does so according to the equilibrium bid function for the situation with no reserve price. This implies that if $B_2(\varphi, v^{-1}(R)) < R$, a bidder with type R would submit a bid below the reserve price, which is not possible, and a contradiction is therefore established.

The condition $R \leq B_2(\varphi, v^{-1}(R))$ is sufficient for the following reason. As said, in a weakly separating Bayesian Nash equilibrium, each bidder who submits a bid does so as if there were no reserve price. Then, for the existence of a weakly separating equilibrium, it remains to be checked that $B_2(\varphi, \hat{t}) \geq R$. If $B_2(\varphi, v^{-1}(R)) \geq R$, then there is a type $\tilde{t} \leq v^{-1}(R)$ for which $B_2(\varphi, \tilde{t}) = R$. As a reserve price does not affect equilibrium bidding of types that submit a bid, it follows that if type \tilde{t} would submit a bid in equilibrium, she would submit a bid equal to R . However, type $v^{-1}(R)$ is indifferent between bidding R and not submitting a bid, so that \tilde{t} prefers not to submit a bid. Therefore, \hat{t} must exceed \tilde{t} , so that indeed $B_2(\varphi, \hat{t}) \geq B_2(\varphi, \tilde{t}) = R$. Observe that \hat{t} jumps to a bid strictly above the reserve price when $B_2(\varphi, \hat{t}) > R$.

The necessary and sufficient condition $R \leq B_2(\varphi, v^{-1}(R))$ implies that only for small R , a weakly separating Bayesian Nash equilibrium exists. As said, the existence of such

an equilibrium is trivial in the case of small R . However, for R close to $v(1)$, $R > B_2(\varphi, v^{-1}(R))$, as, by Proposition 2, $B_2(\varphi, 1) < v(1)$.

If the condition $R \leq B_2(\varphi, v^{-1}(R))$ is violated, then there may be an equilibrium with pooling at R . If $\max\{B_2(\varphi, v^{-1}(R)), B_2(\varphi, 1)\} < R < v(1)$, all types above a threshold L submit a bid equal to the reserve price. Type L is indifferent between bidding and not bidding and follows uniquely from a similar condition as in FPSB:

$$\varphi R (p(1, L) - L^{n-1}) = p(1, L) [v(L) - R]. \quad (9)$$

Moreover, if $R > v(1)$, none of the bidders submit a bid in equilibrium, as the value of winning for none of the bidders exceeds the reserve price.

Proposition 8 *Assume independent private values. Consider SPSB with reserve price R . If $R > \max\{B_2(\varphi, v^{-1}(R)), B_2(\varphi, 1)\}$, then the following bidding strategies constitute a Bayesian Nash equilibrium:*

$$B_2^R(\varphi, t) = \begin{cases} R & \text{if } L < t \leq 1 \\ \text{"no bid"} & \text{if } t \leq L \end{cases}$$

where $L = 1$ if $R > v(1)$, and L follows from (9) otherwise.

The most involved case is $B_2(\varphi, v^{-1}(R)) < R \leq B_2(\varphi, 1)$. If this condition holds, then types below a certain type L abstain from bidding, bidders between types L and H bid the reserve price, and types $t > H$ bid $B_2(\varphi, t)$. The threshold types L and H respectively follow from the following indifference relations:

$$[p(L, H) - L^{n-1}] \varphi R = p(L, H) (v(L) - R) \quad (10)$$

$$[q(H) + r(L, H)] \varphi R = q(H) \varphi B_2(\varphi, H) + r(L, H) (v(H) - R). \quad (11)$$

where $p(L, H)$ is the probability that a bidder wins given that she bids the reserve price, $q(H)$ is the probability that exactly one bidder has a type above H , and $r(L, H)$ is the probability that a bidder does *not* win given that she bids R and that the highest type

of the other bidders does not exceed H . More specifically, $p(L, H)$ is defined in (7),

$$\begin{aligned} q(H) &= (n-1)(1-H)H^{n-2}, \text{ and} \\ r(L, H) &= H^{n-1} - p(L, H). \end{aligned}$$

The indifference relation (10) for type L is constructed as follows. L is indifferent between bidding R and abstaining from bidding. The outcome is different in two situations: first, if the bidder wins against another bidder who bids R (which occurs with probability $p(L, H) - L^{n-1}$), and if no other bidder bids (which happens with probability L^{n-1}).

For type H , the indifference relation (11) follows as she is indifferent between bidding R and submitting a bid $B_2(\varphi, H)$. These two bids result in a different outcome under the following two conditions: first, exactly one bid exceeds R , so that H determines the price of the winner (this event has probability $q(H)$), and second, H , when bidding R , loses against another bidder who also bids R (probability $r(L, H)$).

Proposition 9 *Assume independent private values. Consider SPSB with reserve price R . If $B_2(\varphi, v^{-1}(R)) < R \leq B_2(\varphi, 1)$, the following bidding strategies constitute a Bayesian Nash equilibrium:*

$$B_2^R(\varphi, t) = \begin{cases} B_2(\varphi, t) & \text{if } t > H \\ R & \text{if } L \leq t \leq H \\ \text{"no bid"} & \text{if } t < L \end{cases}$$

where (L, H) is a solution of the system of equations (10) and (11).

To summarize: SPSB has no less than five types of equilibria if the seller requires a minimum bid R . First, if $R < B_2(\varphi, 0)$, all bidders submit a bid according to $B_2(\varphi, t)$. Second, if $R > v(1)$, no bidder bids. Third, if $B_2(\varphi, 1) < R < v(1)$, all bidders above a threshold bid exactly the reserve price. For $R \in (B_2(\varphi, 0), B_2(\varphi, 1))$, the condition $B_2(\varphi, v^{-1}(R)) \geq R$ becomes crucial. If this condition holds true, types up to a threshold do not bid, and types above this threshold submit bids according to $B_2(\varphi, t)$. Otherwise, low types abstain from bidding, intermediate types pool at R , and high types t bid $B_2(\varphi, t)$. All five types of equilibria are shown in Table 1.

Reserve price	Type of equilibrium
$R < B_2(\varphi, 0)$	Separating
$R \in (B_2(\varphi, 0), B_2(\varphi, 1))$ and $R \leq B_2(\varphi, v^{-1}(R))$	Weakly separating
$R \in (B_2(\varphi, 0), B_2(\varphi, 1))$ and $R > B_2(\varphi, v^{-1}(R))$	Pooling at R and high types bid above R
$R \in (B_2(\varphi, 1), v(1))$	Only bids at R
$R > v(1)$	No one bids

Table 1: shape of the equilibrium bid function in SPSB depending on the level of the reserve price.

5 Concluding remarks

We have studied auctions in which losing bidders obtain financial externalities from the winning bidder. We have derived bidding equilibria for FPSB and SPSB, and we have shown that the seller cannot gain more revenue by guaranteeing the losing bidders a fraction of the auction revenue. Additionally, SPSB dominates FPSB in terms of expected auction revenue if $\varphi < \frac{1}{n-1}$ and both auctions are revenue equivalent if $\varphi = \frac{1}{n-1}$. Moreover, we have studied equilibrium bidding for FPSB and SPSB when a reserve price is imposed. We have observed pooling at the reserve price for FPSB, and for SPSB if the reserve price is sufficiently high.

Motivated by the observation that in SPSB, low signal bidders increase their bids when φ is increased (provided that φ is not too large), a model with asymmetries in the valuation function may be fruitful to study. One may imagine that with one bidder with a low value and another with a high value, the price in SPSB may be higher with financial externalities than without financial externalities, as the bidder with the low value has an incentive to push up the price when φ is strictly positive. This indicates that the seller may have an incentive to promise low value bidders a share of the auction revenue. Indeed, Goeree and Offerman (2004) show that in asymmetric environments, the seller may obtain more revenue by rewarding one of the losing bidders a fraction of the auction revenue.

6 Appendix

Throughout the Appendix, we let $F^{[1]}$ [$F^{[2]}$] denote the cumulative distribution function of the first [second] order statistic of $n - 1$ draws from the uniform distribution on $[0, 1]$, and $f^{[1]}$ [$f^{[2]}$] the corresponding density function.

PROOF OF PROPOSITION 1. A higher type of bidder cannot submit a lower bid than a lower type of the same bidder. (If the low type gets the same expected surplus from strategies with two different probabilities of being the winner of the object, the high type strictly prefers the strategy with the highest probability of winning. Therefore, the high type will not submit a lower bid than the low type.) Also, $B_1(\varphi, t)$ cannot be constant on an interval $[t', t'']$. (By bidding slightly higher, a type t'' can largely improve her probability of winning, while only marginally influencing the payments by her and the other bidders.) Moreover, $B_1(\varphi, t)$ cannot be discontinuous at any t . (Suppose that $B_1(\varphi, t)$ makes a jump from \underline{b} to \bar{b} at t^* . A type just above t^* has an incentive to deviate to \underline{b} . Doing so, she is able to decrease the auction price, while just slightly affecting her probability of winning the object. As φ is small enough, she is able to improve her utility.) Hence, a symmetric equilibrium bid function must be strictly increasing and continuous. Define the utility $U(t, s)$ for a bidder with signal t who misrepresents herself as having signal s , whereas the other bidders report truthfully, if the bid function is indeed strictly increasing. Then,

$$U(t, s) = \int_0^s v(t, y) dF^{[1]}(y) - F^{[1]}(s)B_1(\varphi, s) + \varphi \int_s^1 B_1(\varphi, y) dF^{[1]}(y).$$

The first two terms of the RHS of this expression refer to the case that this bidder wins the object. The third term refers to the case that she does not win. Assume that $B_1(\varphi, s)$ is differentiable in s . Maximizing $U(t, s)$ with respect to s and equating s to t gives the FOC of the equilibrium

$$f^{[1]}(t)v(t, t) - f^{[1]}(t)B_1(\varphi, t) - F^{[1]}(t)B_1'(\varphi, t) - \varphi B_1(\varphi, t)f^{[1]}(t) = 0.$$

With some manipulation we get

$$F^{[1]}(t)^\varphi f^{[1]}(t)v(t, t) = (1 + \varphi)B_1(\varphi, t)f^{[1]}(t)F^{[1]}(t)^\varphi + B_1'(\varphi, t)F^{[1]}(t)^{1+\varphi}, \quad (12)$$

or, equivalently,

$$C_1 + \int_0^t F^{[1]}(y)^\varphi f^{[1]}(y)v(y, y)dy = F^{[1]}(t)^{1+\varphi} B_1(\varphi, t),$$

where C_1 is a constant. Substituting $t = 0$ gives $C_1 = 0$, so that the bid function is given by

$$\begin{aligned} B_1(\varphi, t) &= \frac{1}{F^{[1]}(t)} \int_0^t \left(\frac{F^{[1]}(y)}{F^{[1]}(t)} \right)^\varphi f^{[1]}(y)v(y, y)dy \\ &= (n-1)t^{-n+1-\varphi} \int_0^t y^{\varphi(n-1)+n-2}v(y, y)dy. \end{aligned} \quad (13)$$

It is readily checked that the second order condition $\text{sign}\left(\frac{\partial U(t,s)}{\partial s}\right) = \text{sign}(t-s)$ is fulfilled. Using integration by parts, (13) can be rewritten as (3). From (12), we infer that $\frac{\partial B_1(\varphi, t)}{\partial t} > 0$ if and only if $B_1(\varphi, t) < \frac{v(t,t)}{1+\varphi}$, so that indeed $B_1(\varphi, t)$ is strictly increasing in t , as *Value Monotonicity* implies that $\frac{dv(y,y)}{dy} > 0$ for all y . Finally, by *Value Differentiability*, *Value Monotonicity*, and *Symmetry*, the efficiency of the auction outcome is established. Uniqueness follows from a standard argument (see e.g., Bulow et al. 1999). ■

PROOF OF PROPOSITION 2. Following the lines of the proof of Proposition 1 it can be established that a symmetric equilibrium function must be strictly increasing and continuous. The utility for a bidder with signal t acting as if she had signal s is given by

$$U(t, s) = \int_0^s [v(t, y) - B_2(\varphi, y)]dF^{[1]}(y) + \varphi\pi(s)B_2(\varphi, s) + \varphi \int_{y=s}^1 B_2(\varphi, y)dF^{[2]}(y),$$

where $\pi(s) \equiv F^{[2]}(s) - F^{[1]}(s)$ denotes the probability that there is exactly one opponent with a signal larger than s . The first term of the RHS refers to the case that this bidder wins, the second term to the case that she submits the second highest bid, and the third term to her bid being the third or higher. Assume that $B_2(\varphi, s)$ is differentiable in s . The FOC of the equilibrium is

$$[v(t, t) - B_2(\varphi, t)]f^{[1]}(t) + \varphi \frac{\partial \pi(t)B_2(\varphi, t)}{\partial t} - \varphi B_2(\varphi, t)f^{[2]}(t) = 0$$

or, equivalently,

$$v(t, t)f^{[1]}(t) = -B_2'(\varphi, t)\varphi\pi(t) + B_2(\varphi, t)[(1 + \varphi)f^{[1]}(t)]. \quad (14)$$

The general solution to the above differential equation is equal to

$$B_2(\varphi, t)(1 - t)^{\frac{1+\varphi}{\varphi}} = C_2 - \int_0^t (1 - y)^{\frac{1}{\varphi}} v(y, y) dy,$$

where C_2 is a constant. Substituting $t = 1$ yields a unique solution for C_2 :

$$C_2 = \int_0^1 (1 - y)^{\frac{1}{\varphi}} v(y, y) dy.$$

The only possible differentiable bid function that may constitute a symmetric equilibrium is given by

$$B_2(\varphi, t) = \frac{1}{\varphi}(1 - t)^{-1 - \frac{1}{\varphi}} \int_t^1 (1 - y)^{\frac{1}{\varphi}} v(y, y) dy. \quad (15)$$

It is readily checked that the second order condition $\text{sign}\left(\frac{\partial U(t, s)}{\partial s}\right) = \text{sign}(t - s)$ holds. Using integration by parts on $B_2(\varphi, t)$, we see that (15) can also be written as (4). To complete the proof, we must show that $B_2(\varphi, t)$ is indeed increasing in t . From (15), it follows that

$$B_2(\varphi, t) > \frac{v(t, t) \int_t^1 (1 - y)^{\frac{1}{\varphi}} dy}{\varphi(1 - t)^{\frac{1+\varphi}{\varphi}}} = \frac{v(t, t)}{1 + \varphi}.$$

As (14) implies that $B_2'(\varphi, t) > 0$ if and only if $B_2(\varphi, t) > \frac{v(t, t)}{1 + \varphi}$, $B_2(\varphi, t)$ is indeed strictly increasing in t . Then, by *Value Differentiability*, *Value Monotonicity*, and *Symmetry*, it follows that the outcome of the auction is efficient. Uniqueness follows from a standard argument. ■

PROOF OF PROPOSITION 3. As stated in the text, it is sufficient to show that the expected utility $U_2(0)$ of the lowest type is increasing in φ . If $n = 2$ and if the lowest type is present, then the price paid is equal to her bid. Therefore,

$$\begin{aligned} U_2(0) &= \varphi B_2(\varphi, 0) \\ &= \int_0^1 (1 - t)^{\frac{1}{\varphi}} v(t, t) dt. \end{aligned}$$

which is strictly increasing in φ . If $n > 2$, then the bidder receives financial externalities equal to φ times the second highest bid. Therefore, using the expression for the bid function in (15),

$$\begin{aligned}
U_2(0) &= \varphi \int_0^1 B_2(\varphi, t) dF^{[2]}(t) \\
&= (n-1)(n-2) \int_0^1 t^{n-3} (1-t)^{-\frac{1}{\varphi}} \int_t^1 (1-y)^{\frac{1}{\varphi}} v(y, y) dy dt \\
&= (n-1)(n-2) \int_0^1 t^{n-3} \int_t^1 \left(\frac{1-y}{1-t} \right)^{\frac{1}{\varphi}} v(y, y) dy dt,
\end{aligned} \tag{16}$$

which is strictly increasing in φ as in the inner integral, $y > t$. ■

PROOF OF PROPOSITION 4. (The proof follows the same logic as the proof of Proposition 5 in Bulow et al. 1999). Let $U_1(0)$ and $U_2(0)$ be the equilibrium utility of the lowest type in FPSB and SPSB respectively. According to Maasland and Onderstal (2002), for SPSB to generate higher [the same] expected revenue than [as] FPSB, it is sufficient to show that $U_1(0) > U_2(0)$ [$U_1(0) = U_2(0)$]. We split the proof in two cases: $n = 2$ and $n > 2$.

We start with the case $n = 2$. As the outcome of both auctions is efficient, a bidder with type 0 loses the auction with probability 1, and gets financial externalities as the other bidder has to pay. For FPSB, the expected price paid by the other bidder is the expectation of her bid (13). Hence,

$$\begin{aligned}
U_1(0) &= \varphi \int_0^1 B_1(\varphi, t) dt \\
&= \varphi \int_0^1 t^{-1-\varphi} \int_0^t y^\varphi v(y, y) dy dt \\
&= \varphi \int_0^1 y^\varphi v(y, y) \int_y^1 t^{-1-\varphi} dt dy \\
&= \int_0^1 v(y, y) [1 - y^\varphi] dy.
\end{aligned}$$

$U_2(0)$ is already determined in the proof of Proposition 3:

$$U_2(0) = \int_0^1 (1-t)^{\frac{1}{\varphi}} v(t, t) dt.$$

The difference between $U_1(0)$ and $U_2(0)$ can be expressed as

$$U_1(0) - U_2(0) = \int_0^1 \left\{ 1 - t^\varphi - (1-t)^{\frac{1}{\varphi}} \right\} v(t, t) dt.$$

For $\varphi < 1$ [$\varphi = 1$], $U_1(0) - U_2(0) > 0$ [$U_1(0) - U_2(0) = 0$], as the expression in curly brackets has expected value zero, and is strictly negative for all $t \in (0, \hat{t})$ and positive for all $t \in (\hat{t}, 1)$ for some \hat{t} [is zero for all $t \in [0, 1]$].

Let us now consider the case of n bidders. Using the expression for the bid function in (13) and $U_2(0)$ in (16), we derive that

$$\begin{aligned} U_1(0) &= \varphi \int_0^1 B_1(\varphi, t) dF^{[1]}(t) \\ &= (n-1)^2 \varphi \int_0^1 t^{-1-\varphi(n-1)} \int_0^t y^{\varphi(n-1)+n-2} v(y, y) dy dt \\ &= (n-1)^2 \varphi \int_0^1 y^{\varphi(n-1)+n-2} v(y, y) \int_y^1 t^{-1-\varphi(n-1)} dt dy \\ &= (n-1) \int_0^1 [y^{n-2} - y^{\varphi(n-1)+n-2}] v(y, y) dy \end{aligned}$$

and

$$\begin{aligned} U_2(0) &= (n-1)(n-2) \int_0^1 t^{n-3} (1-t)^{-\frac{1}{\varphi}} \int_t^1 (1-y)^{\frac{1}{\varphi}} v(y, y) dy dt \\ &= (n-1)(n-2) \int_0^1 (1-y)^{\frac{1}{\varphi}} v(y, y) \int_0^y t^{n-3} (1-t)^{-\frac{1}{\varphi}} dt dy \\ &= (n-1) \int_0^1 \left\{ y^{n-2} - \frac{1}{\varphi} (1-y)^{\frac{1}{\varphi}} \int_0^y t^{n-2} (1-t)^{-\frac{1}{\varphi}-1} dt \right\} v(y, y) dy. \end{aligned}$$

The difference between $U_1(0)$ and $U_2(0)$ can be expressed as

$$\frac{U_1(0) - U_2(0)}{n-1} = \int_0^1 \left\{ -y^{\varphi(n-1)+n-2} + \frac{1}{\varphi} (1-y)^{\frac{1}{\varphi}} \int_0^y t^{n-2} (1-t)^{-\frac{1}{\varphi}-1} dt \right\} v(y, y) dy.$$

For $\varphi < \frac{1}{n-1}$ [$\varphi = \frac{1}{n-1}$], $U_1(0) - U_2(0) > 0$ [$U_1(0) - U_2(0) = 0$], as the expression in curly brackets has zero expected utility, and is strictly negative for all $y \in (0, \hat{y})$ and positive for all $y \in (\hat{y}, 1)$ for some \hat{y} [is zero for all $y \in [0, 1]$].

The following observations prove the last statement. Let

$$g(\varphi, n, y) \equiv -y^{\varphi(n-1)+n-2} + \frac{1}{\varphi} (1-y)^{\frac{1}{\varphi}} \int_0^y t^{n-2} (1-t)^{-\frac{1}{\varphi}-1} dt.$$

The expectation of $g(\varphi, n, y)$ with respect to y is

$$\int_0^1 g(\varphi, n, y) dy = -\frac{1}{(\varphi+1)(n-1)} + \frac{1}{\varphi} \int_0^1 t^{n-2} (1-t)^{-\frac{1}{\varphi}-1} \int_t^1 (1-y)^{\frac{1}{\varphi}} dy dt = 0.$$

Define

$$\begin{aligned} f(y) &\equiv \varphi (1-y)^{-\frac{1}{\varphi}} g(\varphi, n, y) \\ &= -\varphi (1-y)^{-\frac{1}{\varphi}} y^{\varphi(n-1)+n-2} + \int_0^y t^{n-2} (1-t)^{-\frac{1}{\varphi}-1} dt. \end{aligned}$$

Note that $f(0) = 0$, and $f(y)$ is negative for positive y close to 0, as the first [second] term on the RHS is of the order $y^{\varphi(n-1)+n-2}$ [y^{n-1}] and $\varphi(n-1) + n - 2 < n - 1$. Moreover,

$$f'(y) = -[\varphi(n-1) + n - 2] \varphi (1-y)^{-\frac{1}{\varphi}} y^{\varphi(n-1)+n-3} + y^{n-2} (1-y^{\varphi(n-1)}) (1-y)^{-\frac{1}{\varphi}-1}.$$

Note that $\lim_{y \uparrow 1} f'(y) = +\infty$, and that for $y \in (0, 1)$, $f'(y) = 0$ implies

$$\{y + [\varphi(n-1) + n - 2] \varphi (1-y)\} = y^{1-\varphi(n-1)}.$$

As the function on the LHS is linear, the one on the RHS concave, and the equality holds for $y = 1$, there is at most one point $y \in (0, 1)$ at which $f'(y) = 0$. This implies that f is strictly negative for all $y \in (0, \hat{y})$ and positive for all $y \in (\hat{y}, 1)$ for some \hat{y} , so that the same holds true for g .

For $\varphi = \frac{1}{n-1}$, we prove that $g(\varphi = \frac{1}{n-1}, n, y) = 0$ for all $y \in [0, 1]$ by induction to n . It is straightforwardly checked that $g(\varphi = 1, 2, y) = 0$. Suppose that $g(\varphi = \frac{1}{n-1}, n, y) = 0$ for some $n \geq 2$. We now show that this implies that $g(\varphi = \frac{1}{n}, n+1, y) = 0$:

$$\begin{aligned}
g\left(\frac{1}{n}, n+1, y\right) &= -y^n + n(1-y)^n \int_0^y t^{n-1}(1-t)^{-n-1} dt \\
&= -y^n + y^{n-1} - (n-1)(1-y)^n \int_0^y t^{n-2}(1-t)^{-n} dt \\
&= -y^n + y^{n-1} - (1-y) \left[g\left(\frac{1}{n-1}, n, y\right) + y^{n-1} \right] \\
&= 0.
\end{aligned}$$

■

PROOF OF PROPOSITION 5. Assume that threshold types L and H exist such that in equilibrium all types $t < L$ abstain from bidding, all types $t \in [L, H]$ bid R , and all types $t > H$ bid according to a strictly increasing bid function g^R .

$p(L, H)$ is constructed as follows:

$$\begin{aligned}
p(L, H) &= \sum_{i=0}^{n-1} \frac{1}{i+1} \binom{n-1}{i} L^{n-1-i} (H-L)^i \\
&= \frac{1}{n(H-L)} \sum_{j=1}^n \binom{n}{j} L^{n-j} (H-L)^j \\
&= \frac{1}{n(H-L)} (H^n - L^n).
\end{aligned}$$

For type L , the indifference relation is

$$\begin{aligned}
\varphi R (p(H, L) - L^{n-1}) &= p(H, L) [v(L) - R] \implies \\
1 - \frac{L^{n-1}}{p(H, L)} &= \frac{v(L) - R}{\varphi R} \implies \\
1 - \frac{n(HL^{n-1} - L^n)}{H^n - L^n} &= \frac{v(L) - R}{\varphi R} \implies \\
1 - n - \frac{n(HL^{n-1} - H^n)}{H^n - L^n} &= \frac{v(L) - R}{\varphi R}. \tag{17}
\end{aligned}$$

L is uniquely determined from (17) as the LHS of (17) is strictly decreasing in L and the RHS is strictly increasing in L for $L \geq 0$.

A type H is indifferent between bidding R and bidding an infinitesimal δ above R . These two bids only yield a different outcome if all other types are lower than H . The difference between bidding R and a bid just above R is that in the former case the bidder always wins and gains $v(H) - R$, whereas a bid equal to R may result in utility φR if another bidder also bids R . Hence, H is indifferent if and only if

$$v(H) = (1 + \varphi)R.$$

In order to complete the proof, we need to check whether types have no incentive to deviate from the proposed equilibrium. We only check if a type $t > H$ has no incentive to mimic another type $t' > H$, as by a standard argument, other deviations are not profitable. Incentive compatibility of types $t > H$ implies that g^R should follow from the same differential equation as derived in the proof of Proposition 1 with the boundary condition $g^R(H) = R$. This is indeed how g^R is constructed.

Finally, we should establish that $g^R(t)$ is strictly increasing for $t \geq H$. Analogous to the proof of Proposition 1, this is the case if and only if $g^R(t) < \frac{v(t)}{1+\varphi}$ for almost all $t \in [H, 1]$. Now,

$$\begin{aligned} g^R(t) &= (n-1)t^{-(n-1)(1+\varphi)} \left[\int_H^t y^{(n-1)\varphi+n-2} v(y) dy + v(H)H^{(n-1)(1+\varphi)} \right] \\ &= \frac{v(t)}{1+\varphi} - \frac{t^{-(n-1)(1+\varphi)}}{1+\varphi} \int_H^t y^{(n-1)(\varphi+1)} v'(y) dy \\ &< \frac{v(t)}{1+\varphi}. \end{aligned}$$

■

PROOF OF PROPOSITION 6. The proof is by contradiction. Suppose that a weakly separating equilibrium does exist. Then all types above a threshold type \hat{t} submit a bid according to a strictly increasing bid function, which we denote by β . Then it must be the case that $\beta(\hat{t}) = R$ (a bid strictly below R is not allowed, and it cannot be the case in equilibrium that $\beta(\hat{t}) > R$, as \hat{t} would strictly prefer to deviate to a bid of R). Moreover, type \hat{t} is indifferent between bidding R and not bidding, which only makes a difference if

no other bidder bids. Therefore, the indifference relation for \hat{t} is:

$$0 = v(\hat{t}) - R$$

so that $\hat{t} = v^{-1}(R)$. Now, analogously to the proof of Proposition 1, the utility of a type t that wishes to mimic a type $s > \hat{t}$ is given by

$$U(t, s) = \int_{\hat{t}}^s v(t) dF^{[1]}(y) - F^{[1]}(s)\beta(s) + \varphi \int_s^1 \beta(y) dF^{[1]}(y).$$

The equilibrium bid function follows from the condition

$$\frac{\partial U(t, s)}{\partial s} = 0$$

at $s = t$, which results in the following differential equation:

$$F^{[1]}(t)^\varphi f^{[1]}(t)v(t) = (1 + \varphi)\beta(t)f^{[1]}(t)F^{[1]}(t)^\varphi + \beta'(t)F^{[1]}(t)^{1+\varphi}. \quad (18)$$

Then a necessary condition for β to be strictly increasing (the assumption we started with) is $\beta'(t) \geq 0$ for all $t \geq \hat{t}$. (18) implies that this condition is equivalent to $\beta(t) \leq \frac{v(t)}{1+\varphi}$, so that it must be true that

$$\beta(\hat{t}) \leq \frac{v(\hat{t})}{1 + \varphi} = \frac{R}{1 + \varphi} < R.$$

In other words, a weakly separating equilibrium can only exist if type \hat{t} submits a bid strictly below R . However, this contradicts the fact that all submitted bids should exceed the reserve price R . ■

PROOF OF PROPOSITIONS 7 - 9. Suppose that all types t above a threshold type \hat{t} submit a bid above R according to a strictly increasing bid function g . Analogous to the proof of Proposition 2, the utility of a type t that wishes to mimic a type $s > \hat{t}$ is given by

$$U(t, s) = \int_{\hat{t}}^s [v(t) - g(y)] dF^{[1]}(y) + \varphi \pi(s)g(s) + \varphi \int_{y=s}^1 g(y) dF^{[2]}(y).$$

The equilibrium bid function follows from the condition

$$\frac{\partial U(t, s)}{\partial s} = 0$$

at $s = t$. This immediately leads to the same differential equation as derived in the proof of Proposition 2 with the same boundary condition $g(1) = \frac{v(1)}{1+\varphi}$, so that $B_2^R(\varphi, t) = B_2(\varphi, t)$ is a solution for all R and $t \geq \hat{t}$.

Now, suppose there is an R for which a weakly separating equilibrium exists. Then there is an indifference type \hat{t} such that

$$B_2^R(\varphi, t) = \begin{cases} B_2(\varphi, t) & \text{if } t \geq \hat{t} \\ \text{"no bid"} & \text{if } t < \hat{t} \end{cases}$$

is an equilibrium. \hat{t} is indifferent between submitting no bid, and submitting a bid equal to $B_2(\varphi, \hat{t})$, so that indeed \hat{t} follows from (8). By the intermediate value theorem, (8) has a solution as for $\hat{t} = 0$ [$\hat{t} = 1$], the LHS is smaller [larger] than the RHS. Moreover, observe that (8) can be rewritten as

$$\varphi B_2(\varphi, \hat{t}) + \frac{\hat{t}^n}{1 - \hat{t}^n} [v(\hat{t}) - R] = \frac{\varphi(1 - \hat{t})R}{1 - \hat{t}^n}. \quad (19)$$

(19) has a unique solution as the RHS [LHS] is strictly increasing [decreasing] in \hat{t} .

A weakly separating equilibrium exists if and only if $B_2(\varphi, \hat{t}) \geq R$, as all bids should be above R . We will now show that $B_2(\varphi, \hat{t}) \geq R$ is equivalent to the condition $B_2(\varphi, v^{-1}(R)) \geq R$. Define \tilde{t} such that

$$B_2(\varphi, \tilde{t}) = R. \quad (20)$$

As $B_2(\varphi, t)$ is strictly increasing in t , \tilde{t} is uniquely determined. Consider the function h with

$$h(t) \equiv \varphi B_2(\varphi, t) + \frac{t^n [v(t) - R]}{1 - t^n}$$

for all t . Note that h is a strictly increasing function, with

$$h(\hat{t}) = \varphi R,$$

(which follows from (8)), and

$$h(\tilde{t}) = \varphi R + \frac{\tilde{t}^n [v(\tilde{t}) - R]}{1 - \tilde{t}^n}.$$

Now, as $B_2(\varphi, t)$, $h(t)$, and $v(t)$ are strictly increasing in t ,

$$\begin{aligned} B_2(\varphi, \hat{t}) \geq R = B_2(\varphi, \tilde{t}) &\iff \hat{t} \geq \tilde{t} \iff h(\hat{t}) \geq h(\tilde{t}) \iff R \geq v(\tilde{t}) \iff \\ v^{-1}(R) \geq \tilde{t} &\iff B_2(\varphi, v^{-1}(R)) \geq B_2(\varphi, \tilde{t}) = R. \end{aligned}$$

Finally, if $B_2(\varphi, v^{-1}(R)) < R$, the pooling equilibrium is straightforwardly established. What remains to be checked is whether the system of equations (10) and (11) has a solution. First, we fix $H \geq v^{-1}(R)$, so that L is a solution of

$$\Phi(x) \equiv [p(x, H) - x^{n-1}] \varphi R - p(x, H) (v(x) - R) = 0. \quad (21)$$

Note that Φ is strictly decreasing. Moreover, $\Phi(0) > 0$ and $\Phi(H) < 0$, so that (21) has a unique solution $L(H) \in (0, H)$. By the implicit function theorem, L is a continuous function.

Now, we check that the following equation has a solution $y \geq v^{-1}(R)$:

$$\Psi(y) \equiv [q(y) + r(L(y), y)] \varphi R - q(y) \varphi B_2(\varphi, y) - r(L(y), y) (v(y) - R) = 0.$$

Observe that $L(v^{-1}(R)) = v^{-1}(R)$, so that $\Psi(v^{-1}(R)) > 0$ (as $r(H, H) = 0$ for all H , and $B_2(\varphi, v^{-1}(R)) < v^{-1}(R)$). Furthermore, let $\tilde{x} \equiv v^{-1}((1 + \varphi)R)$. Note that $\tilde{x} \in (v^{-1}(R), 1)$ and $B_2(\varphi, \tilde{x}) \geq \frac{1}{1+\varphi}v(\tilde{x}) = R$.¹⁵ Then, $\Psi(\tilde{x}) = q(\tilde{x})\varphi [R - B_2(\varphi, \tilde{x})] \leq 0$. Hence, by the intermediate value theorem, there is a $y \in [v^{-1}(R), \tilde{x}]$ for which $\Psi(y) = 0$.¹⁶

■

¹⁵ $\tilde{x} < 1$ as $v(\tilde{x}) = (1 + \varphi)R < (1 + \varphi)B_2(\varphi, 1) = v(1)$.

¹⁶The solution is not necessarily unique.

7 References

- Angeles de Frutos, M.: Asymmetric price-benefits auctions. *Games and Economic Behavior* **33**, 48-71 (2000)
- Benoît, J.-P., and Krishna, V.: Multiple-object auctions with budget constrained bidders. *Review of Economic Studies* **68**, 155-179 (2001)
- Börgers, T., and Dustmann, C.: Strange bids: bidding behaviour in the United Kingdom's third generation spectrum auction. *Economic Journal* (forthcoming)
- Bulow, J., Huang, M., and Klemperer, P.: Toeholds and takeovers. *Journal of Political Economy* **107**, 427-454 (1999)
- Comanor, W. S., and Schankerman, M. A.: Identical bids and cartel behavior. *Bell Journal of Economics* **7**, 281-286 (1976)
- Cramton, P., Gibbons, R., and Klemperer, P.: Dissolving a partnership efficiently. *Econometrica* **55**, 615-632 (1987)
- Dasgupta, S., and Tsui, K.: Auctions with cross-shareholdings. *Economic Theory* **24**, 163-194 (2004)
- Das Varma, G.: Standard auctions with identity-dependent externalities. *Rand Journal of Economics* **33**, 689-708 (2002)
- Deltas, G.: Determining damages from the operation of bidding rings: an analysis of the post-auction 'knockout' sale. *Economic Theory* **19**, 243-269 (2002)
- Engelbrecht-Wiggans, R.: Auctions with price-proportional benefits to bidders. *Games and Economic Behavior* **6**, 339-346 (1994)
- Engers, M., and McManus, B.: Charity auctions. Mimeo, University of Virginia (2004)
- Ettinger, D.: Bidding among friends and enemies. FEEM Working Paper No. 23.2003 (2003a)
- Ettinger, D.: Efficiency in auctions with crossholdings. *Economics Letters* **80**, 1-7 (2003b)
- Ettinger, D.: Takeovers, toeholds, and deterrence. Mimeo, University College London (2003c)
- Goeree, J. K., Maasland, E., Onderstal, S., and Turner, J. L.: How (not) to raise

money. *Journal of Political Economy* (forthcoming)

Goeree, J. K., and Offerman, T.: The Amsterdam auction. *Econometrica* **72**, 281-294 (2004)

Graham, D. M., and Marshall, R. C.: Collusive bidder behavior at single-object second-price and English auctions. *Journal of Political Economy* **95**, 1217-1239 (1987)

Jehiel, P., and Moldovanu, B.: Strategic nonparticipation. *Rand Journal of Economics* **27**, 84-98 (1996)

Jehiel, P., and Moldovanu, B.: Auctions with downstream interaction among buyers. *Rand Journal of Economics* **31**, 768-791 (2000)

Jehiel, P., Moldovanu, B., and Stacchetti, E.: How (not) to sell nuclear weapons. *American Economic Review* **86**, 814-829 (1996)

Jehiel, P., Moldovanu, B., and Stacchetti, E.: Multidimensional mechanism design for auctions with externalities. *Journal of Economic Theory* **85**, 258-283 (1999)

Kittsteiner, T.: Partnerships and double auctions with interdependent valuations. *Games and Economic Behavior* **44**, 54-76 (2003)

Maasland, E., and Onderstal, S.: Optimal auctions with financial externalities. CenterER discussion paper 2002-21, Tilburg University (2002)

McAfee, R. P., and McMillan, J.: Bidding rings. *American Economic Review* **82**, 579-599 (1992)

Milgrom, P. R., and Weber, R. J.: A theory of auctions and competitive bidding. *Econometrica* **50**, 1089-1122 (1982)

Morgan, J.: Dissolving a partnership (un)fairly. *Economic Theory* **23**, 909-923 (2004)

Myerson, R. B.: Optimal auction design. *Mathematics of Operations Research* **6**, 58-73 (1981)

United States v. Chas. Pfizer Co., Inc., 217 F. Supp. 199 (1963)

Van Damme, E.: Fair division under asymmetric information. In: R. Selten (ed.), *Rational interaction - essays in honor of John C. Harsanyi*. Berlin Heidelberg: Springer, 121-144 (1992)